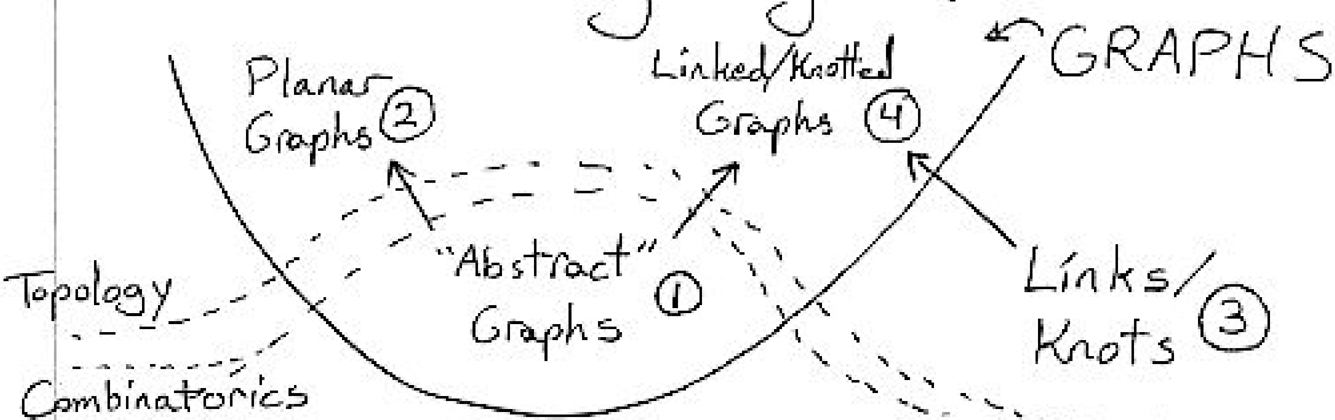
# An Introduction to Knotted and Linked Graphs

This introduction ~~assumes that you the reader~~ is written for the mythical "educated layperson" who may not have taken any proof-based mathematics classes in college, but is "reasonably smart." Our aim will be to explain all of the terms and significance of the following diagram:




~~Ther~~ Ultimately, we will introduce the questions: When is a graph G Planar? Linked? Knotted?

We will begin by introducing Abstract Graphs 1, and then examine the relatively simple question, "when is a graph planar?" 2. From there we will discuss mathematical knots and links 3 and show how the study of linked and knotted graphs resembles the study of planar graphs 4.

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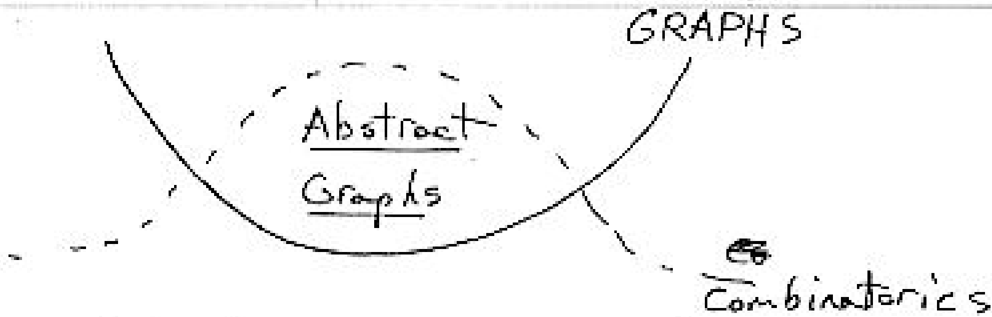
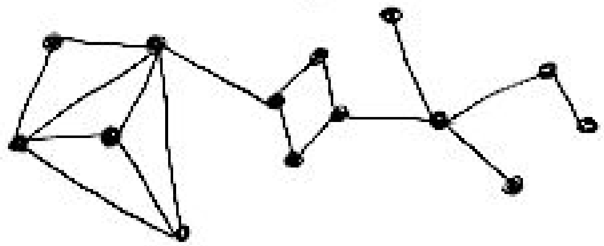



Combinatorics studies the way we put together and construct things, abstractly. A graph is one sort of combinatorial object, composed out of nodes/vertices/points connected together by edges. Here is a picture of a graph:



However, a graph doesn't have to have a picture. Suppose we wanted to make a graph representing ~~friend~~ "friends" on Facebook.

The Facebook Friend Graph:

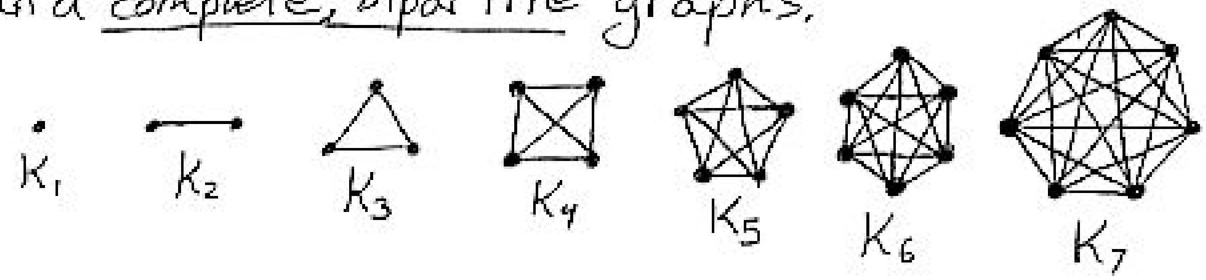
Let every user on Facebook be a node.
Then "draw" an edge between every pair of friends.

Here are some other examples of graphs:

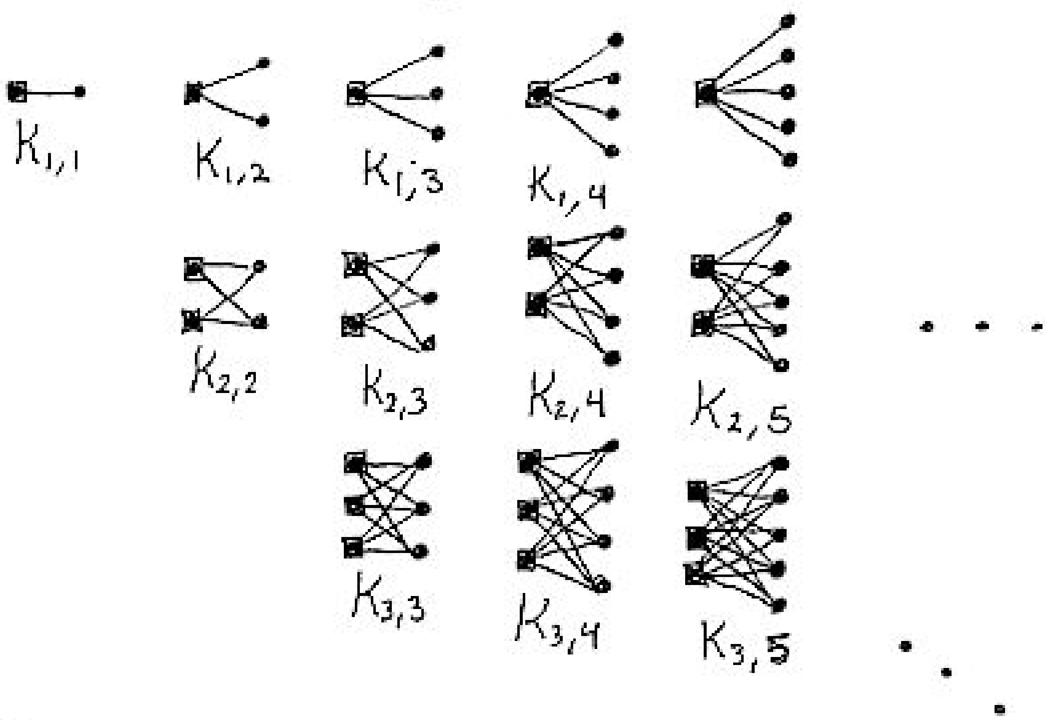
- Internet Servers and connections between them
- Airports and flights
- Cities and roads

(I believe Biologists and Epidemiologists study empirical graphs under the name Network Theory.)

Two very important "families" of graphs that we will refer back to are the complete and complete, bipartite graphs.




The complete graph on $n$ vertices, denoted $K_n$, is a graph with $n$ vertices and all possible edges connecting those $n$ vertices.




The complete bipartite graph on $n$ and $m$ vertices, denoted $K_{n,m}$ is a graph with $n+m$ vertices, ~~n of which are~~ in two groups ($n$ "boxes" and $m$ "circles") and all edges connecting two members of different groups.

(Note that $K_{n,m} = K_{m,n}$ since circles/boxes are arbitrary)

Although graphs have been studied since (at least) Euler's famous Königsberg bridge problem, they have only found widespread applications in the last 50-60 years, largely thanks to prosthelytization by computer scientists.

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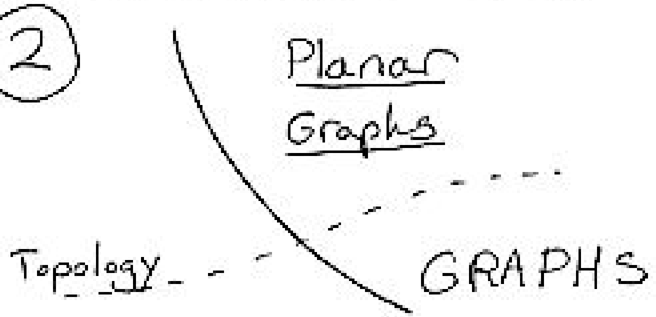



Now, we'll look at graphs topologically.

Topology studies the way things are connected. It is often contrasted with Geometry, earning the monicker "Rubber sheet Geometry"

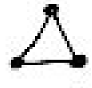
In Geometry, we can rotate and translate our objects without changing them. In Topology, we can go further, to ~~and~~ bend, stretch, squish and generally deform our objects, but we can't cut, break or smash them.

Thus, a square, triangle, and circle are all the "same" topological object, even though they are geometrically distinct.

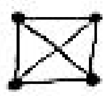
One simple, topological question we can ask about graphs is:

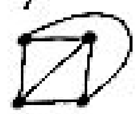
Can we draw a graph $G$, in the plane, such that no two edges ever cross?

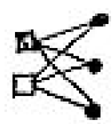
If we can, we call the graph $G$ <u>planar</u>, and if not, we call it <u>non-planar</u>.

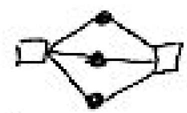
Clearly, the graph $K_3$  is planar.

Perhaps less obviously $K_4$ and $K_{2,3}$ are both planar.


$K_4$


also $K_4$

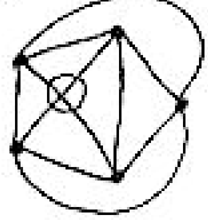

$K_{2,3}$

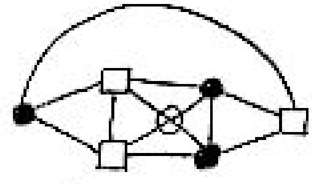

also $K_{2,3}$

Notice that $K_4$ is planar because there is <u>some</u> planar drawing. It is a much harder matter to prove that a graph is non-planar.

<u>Theorem</u>: The graphs $K_5$ and $K_{3,3}$ are non-planar.

Although we will not prove this theorem, you can convince yourself that it is plausible by trying to draw planar diagrams of $K_5$ and $K_{3,3}$. For instance,

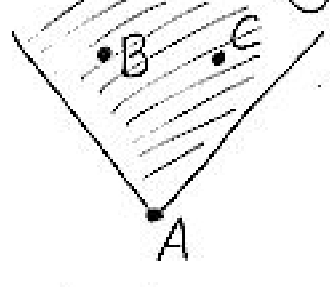

$K_5$ with one crossing


$K_{3,3}$ with one crossing

In 1930, the Polish mathematician Kazimierz Kuratowski showed, rather surprisingly, that these two graphs ($K_5$ and $K_{3,3}$) completely characterize the planarity/non-planarity property in the following way:

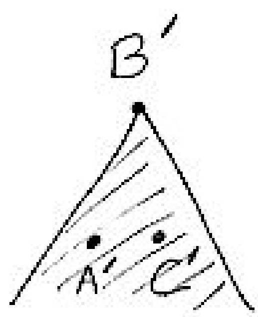
<u>Kuratowski's Theorem:</u> A graph G is non-planar if and only if it "contains" $K_5$ or $K_{3,3}$.

In order to precisely understand the statement that a graph B "contains" a graph A, one must meditate on the topic for a while[End note]. For present purposes it suffices to get the flavor of the concept. We write $A \leq B$ to mean A is "contained" in B, and draw the following conceptual diagrams:




$A \leq B$

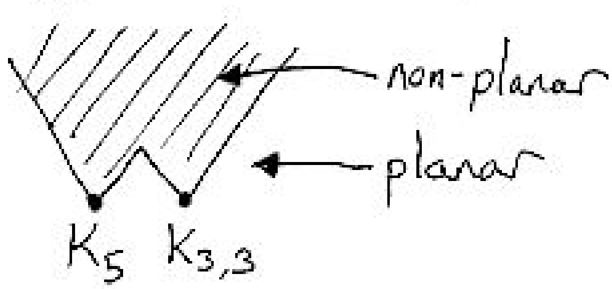
$A \leq C$




$A' \leq B'$

$C' \leq B'$

Thus, Kuratowski's theorem is visualized as



What is significant about Kuratowski's theorem? Well, (as some mathematicians say: "Morally speaking") we are deriving an "intrinsic" property of abstract graphs based on how we can "realize" them topologically.

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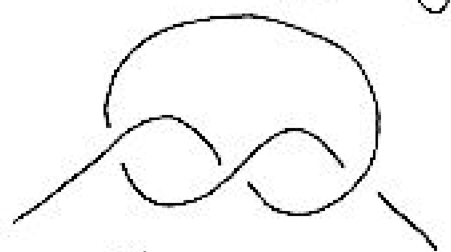
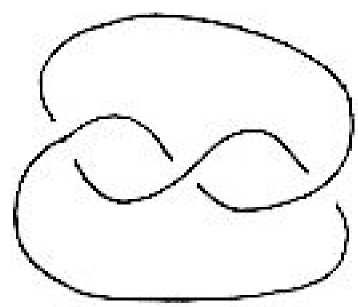
Links & Knots

Topology

Colloquially, we think of a knot as being tied in a piece of string. e.g.

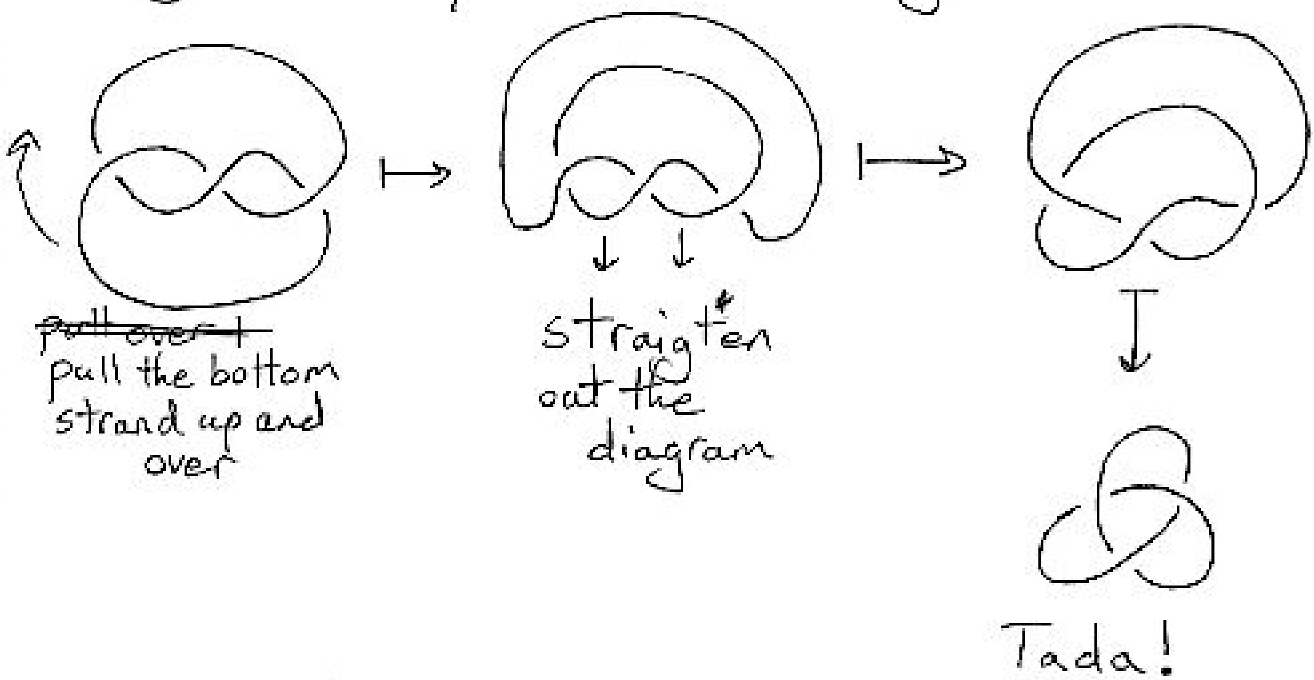


However, mathematically, we require that the string forms a closed loop. This way we can't just untie the knot.

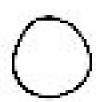


The Trefoil Knot

Because knots are topological objects, we can bend and twist them as we please. By pulling the bottom strand of the trefoil up, we get a very different diagram.

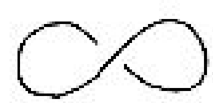



This is also the trefoil. Here are two other knots:



The Unknot (or "trivial" knot)

The Figure Eight Knot



The Unknot (Again!)

Sometimes we can have very complicated pictures of very simple knots. Here is a nasty picture of the unknot:

Therefore, the most natural question in knot theory is: When are two knot diagrams actually diagrams of the same knot? Or even more simply, when is a knot ~~diagram~~ actually knotted? (ie. not the unknot)

For the time being, we will skip over this question.

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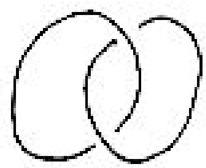
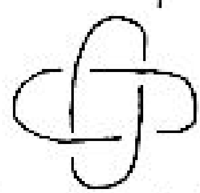
Once you know what a knot is, links are fairly straight-forward. Sometimes a diagram will use more than one piece of string.



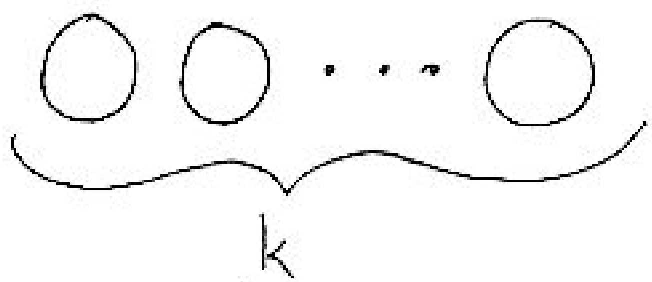
The Hopf Link



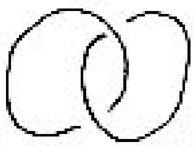
The Whitehead Link

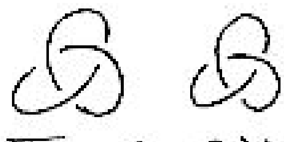
In such a case, we call the depicted object a <u>link</u>. When k loops are used, we call the object a <u>k-component link</u>.

For instance, here is the unlink (aka. the "trivial" link) on k components



Given a 2-component link, we say that ~~the object is linked~~ the two components, that is the two loops, the two cycles, are <u>linked</u> if we cannot disentangle them into two disjoint knots. For instance,


The Hopf Link


Two trefoils

the two cycles of the Hopf link are linked, whereas the two trefoils are not.

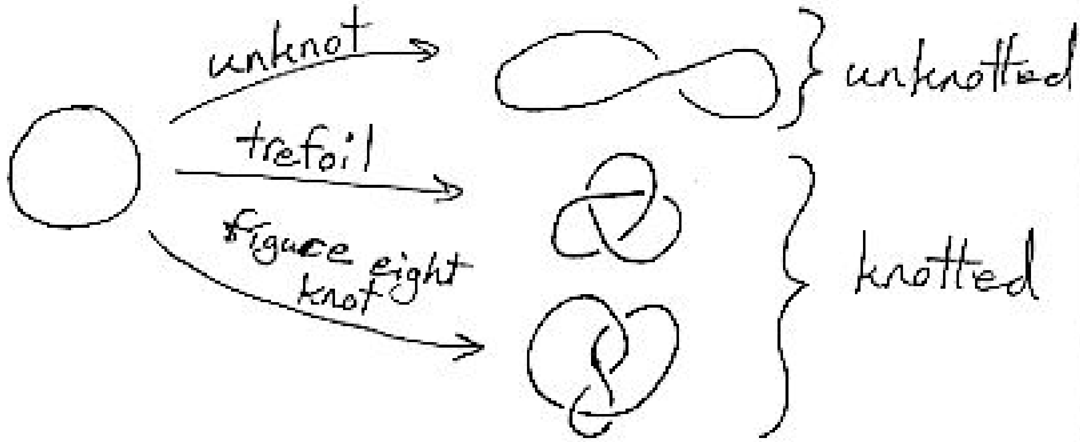
(Note that the two trefoils are not equivalent to the 2-component unlink. Thus, not all unlinked links are trivial)

It is a relatively simple matter to check whether a diagram is linked [End note], although we will avoid that digression.

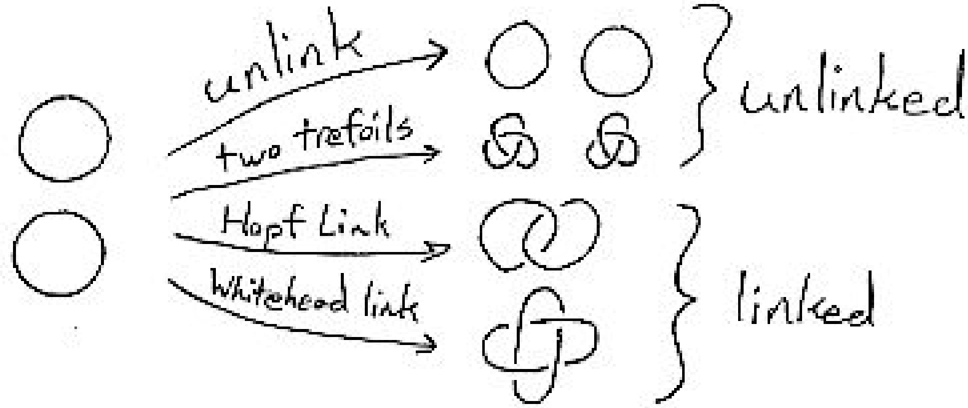
Instead, we will look at a different perspective on what a knot or link is.

We have been thinking of a knot/~~link~~ as an object in 3 dimensional space. However, all of these different objects, these different knots, are the same in another sense. They are all circles/loops/cycles.

Taking this perspective we can think of every knot as some <u>embedding</u> of "the" (canonical) circle/loop/cycle into space.




Similarly, we can think of every 2-component link as some <u>embedding</u> of two canonical circles/loops/cycles into space.



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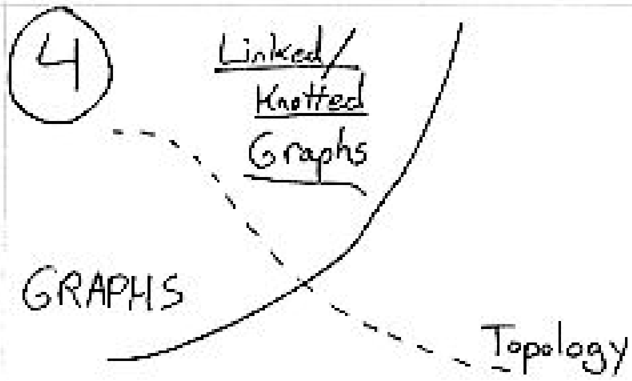
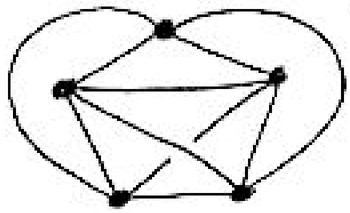



Besides embedding circles (knots) or pairs of circles (links) into space, we can also embed graphs!

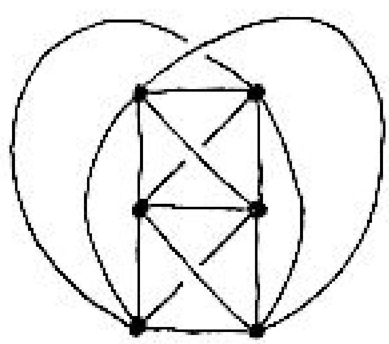
For instance, we couldn't "embed" $K_5$ into the plane, but there's plenty of space in 3d.



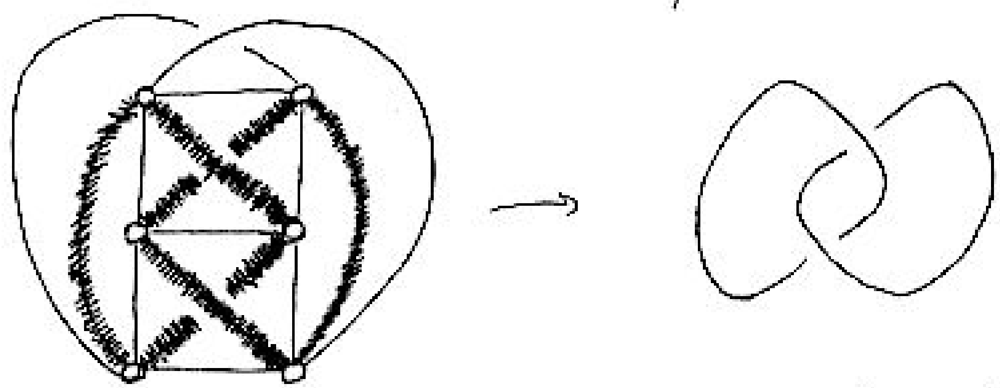
$K_5$ in space.

In fact, it's easy to see that any graph, no matter how big and complicated, can be embedded into 3 dimensions.
(Unlike the plane, which only admits embeddings of planar graphs)

It's far less clear whether every graph can be embedded into space in an "unknotted" or "unlinked" way. It's not even immediately clear what a knotted/linked graph is.
Consider the following diagram of $K_6$:



Contained inside it is a Hopf link!



This embedding of $K_6$ contains a link, but maybe we can find some other <u>linkless embedding</u>. Actually, for $K_6$ we can't. ~~Every emb~~

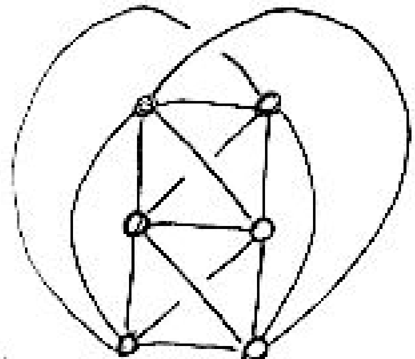
<u>Theorem</u>: Every embedding of $K_6$ contains at least one pair of linked cycles.

We therefore say that $K_6$ is <u>intrinsically linked</u> (abbreviated IL sometimes), and similarly with other graphs. If on the other hand, we can find some embedding of a graph G where every pair of cycles is unlinked, then we say G is <u>not intrinsically linked</u> (NIL)

~~Now the con~~ We may define the notion of a "knotted" graph similarly. If every embedding of a graph G contains at least one knotted cycle, then we call the graph <u>intrinsically knotted</u> (IK).

Conversely, if we can find some embedding of the graph G where every cycle is ~~unknotted~~, unknotted, then we say that G is <u>not intrinsically knotted</u> (NIK).

As an example, consider $K_6$. In this embedding

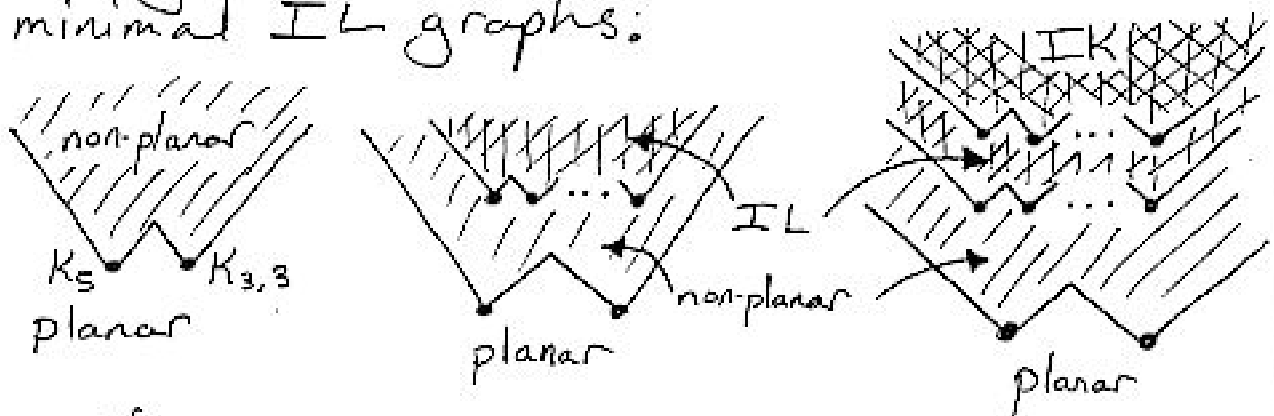


there are no knotted cycles. (though there is a pair of linked cycles) Therefore, we know that $K_6$ is NIK (not intrinsically knotted). By way of contrast,

<u>Theorem:</u> $K_7$ is IK (intrinsically knotted).

In all of these cases, we have said "intrinsic" to emphasize that IK/NIK and IL/NIL are properties of the abstract graph G. As we did with planarity, we are again using topological realizations of graphs to derive intrinsic properties of the abstract graphs.

In fact the properties of non-planarity, intrinsic linking, and intrinsic knotting are all very similar. ~~...~~ In the same way that non-planarity is characterized by two minimal graphs ($K_5$ & $K_{3,3}$), ~~...~~ intrinsic linking is characterized by a set of minimal IL graphs:




Furthermore it can be shown that

<u>Theorem</u>: Every IK graph is necessarily IL.

which justifies our final conceptual picture depicting three strata of classification. Setting the final stone we call the set of minimal graphs for a property (like IL, IK) an <u>obstruction set</u>.

## <u>Our Problem</u>

What is the obstruction set for the intrinsic knotting property?

# Progress on this and related problems

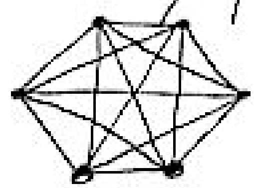
~~(specifically IL)~~

The complete ~~set of~~ obstruction set is known for the IL/NIL property.

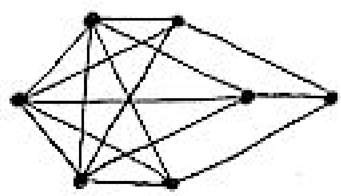
**Theorem** (Conjecture by Sachs in 1981; proof by Robertson, Seymour and Thomas in 1995)

~~The obstruction set for~~
A graph G is intrinsically linked if and only if it "contains" some graph in the Petersen Family of graphs.

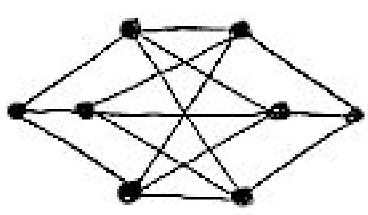
The Petersen Family portrait



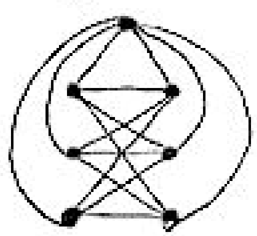
$K_6$



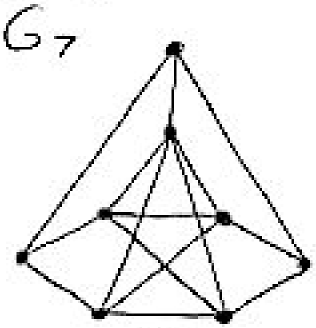
$G_7$



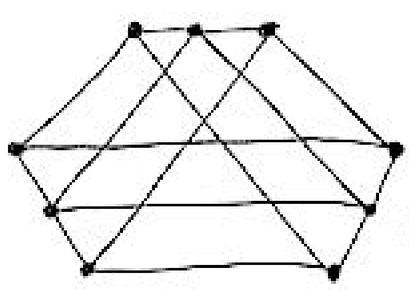
$K_{4,4}^-$



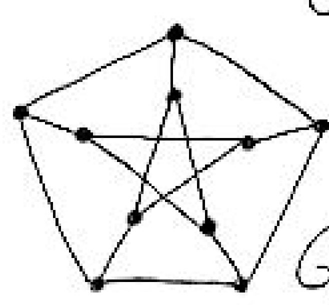
$K_{3,3,1}$



$G_8$



$G_9$



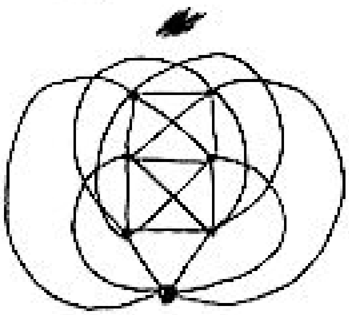
$G_{10}$

$G_{10}$ is often called <u>the Petersen Graph</u>.

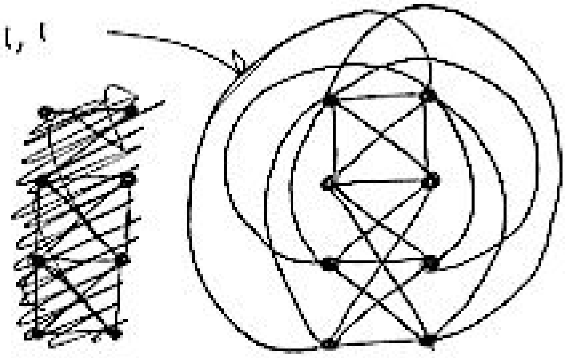
The obstruction set for IK, by contrast, is only partially known

<u>Theorems.</u> A graph $G$ is intrinsically knotted if it "contains" $K_7$, $K_{3,3,1,1}$, or $H$. Furthermore any graph "contained" in one of these three (but not one of these three) is not intrinsically knotted.

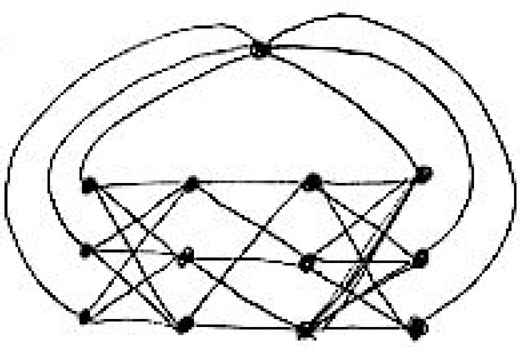
$K_7$



$K_{3,3,1,1}$



$H$



## Endnote #1 on a graph A "containing" a graph B

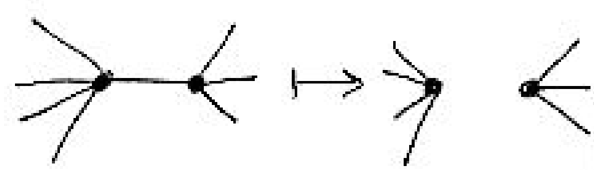
Recall that Kuratowski's Theorem said that a graph is non-planar if and only if it "contains" $K_5$ or $K_{3,3}$. The fully rigorous statement is.

**Kuratowski's Theorem (As stated by Wagner)**

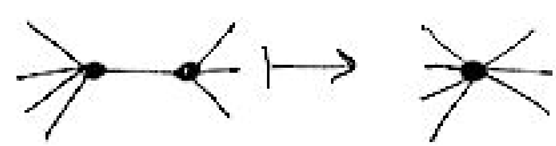
A graph $G$ is non-planar if and only if $K_5$ or $K_{3,3}$ is a minor of $G$.

**Definition** A graph $M$ is a minor of a graph $G$, written $M \leq G$, if we can reduce $G$ to $M$ by

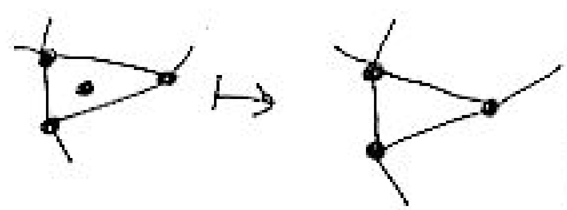
(1) deleting edges



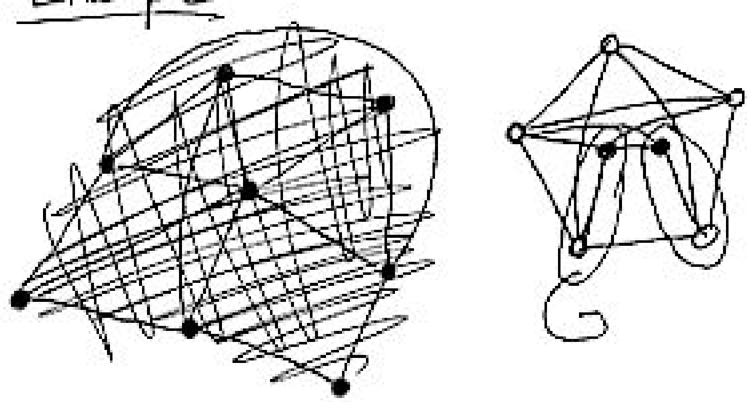
(2) contracting edges



(3) removing isolated vertices



**Example**



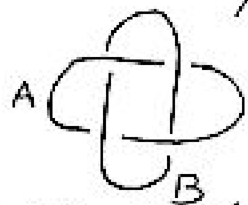

This graph is non-planar because we can contract the two black vertices into the two bottom vertices to yield $K_5$:

$$K_5 \leq G$$

## Endnote #2 on checking whether a diagram is linked

In order to determine whether two components in a diagram are linked, we will count how many times they are linked. For instance,

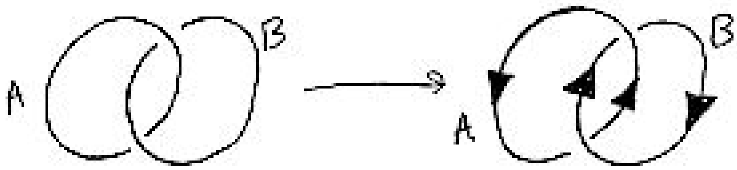

Hopf Link




Whitehead Link

the Hopf Link is linked once, whereas the Whitehead link is linked twice. In order to compute this <u>linking number</u> we will first compute something called the <u>writhe</u>.

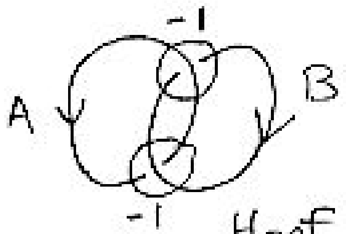
Given a diagram with two components, construct an <u>oriented diagram</u> by drawing little arrows on each component, thus "orienting" it.



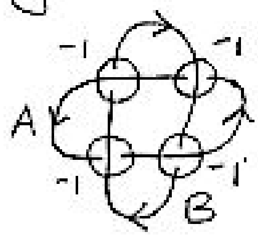

Once we have an oriented diagram, we can distinguish between "positive and "negative" crossings in the following way:

$\varepsilon = -1$ $\qquad$ $\varepsilon = +1$

Here are annotated diagrams of our two links:



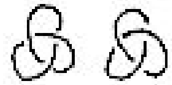
Hopf



Whitehead

We define the <u>writhe</u> of an oriented diagram as the sum of these numbers across all crossing between the two different components.

$$w(A,B) = \sum \varepsilon$$

(Note that there are no crossings between, say, the two trefoils  because the two different components never cross with each other, though they do cross with themselves.)

Therefore we compute that the writhe of the Hopf link diagram above is -2 and -4 for the Whitehead link diagram. In fact, this number will always be even, and the sign will vary depending on our choice of orientation, so we take the absolute value over two.

<u>Definition</u> The linking number between two components, A & B of a diagram is

$$lk(A,B) = \frac{|w(A,B)|}{2}$$

We say a diagram is linked if it has a non-zero linking number.